$$C_2 = -K_{1e}\left(3I_y/2 - \frac{\nu_{zy}G_{xz}}{E}I_x - x_3M_y\right)Y_{31} + K_{1e}(I_{xy} - y_3M_y)X_{31} + K_{2e}\left(3I_x/2 - \frac{\nu_{zx}G_{yz}}{E} \times I_y - y_3M_x\right)X_{31} - K_{2e}(I_{xy} - x_3M_x)Y_{31}$$

$$C_3 = -K_{1e}\left(3I_y/2 - \frac{\nu_{zy}G_{xz}}{E}I_y - x_1M_y\right)Y_{12} + K_{1e}(I_{xy} - y_1M_y)X_{12} + K_{2e}\left(3I_x/2 - \frac{\nu_{zx}G_{yz}}{E} \times I_y - y_1M_x\right)X_{12} - K_{2e}(I_{xy} - x_1M_x)Y_{12}$$

The quantities M_x, M_y, I_x, I_y, I_{xy} refer to the area properties of the element. Having the stiffness matrix for the triangular element, a stiffness matrix for the arbitrary quadrilateral element of Fig. 4, Ref. 9, may be derived as indicated in that reference. Note that the bending and torsion matrices differ only in the vector C_i.

The procedure for solution of problems using this matrix, i.e., the derivation of any "problem matrix" and its solution, has been given in Ref. 9. Inhomogeneity is achieved by using different moduli of elasticity in appropriate elements.

4. Numerical Examples

The problem of a sandwich beam of four layers was analyzed. The inner layers had a modulus which was three times those of the two outer layers although the Poisson's ratio was the same in both. The shear stresses, parallel and perpendicular to the load, obtained for a 256-element break up were within 3% of those obtained from an exact stress function solution.

The next problem considered was that of a load applied along a diameter of a circular section of constant Poisson's ratio, and Young's modulus which varied in proportion to the absolute value of the distance perpendicular to the plane of loading. In the finite element method each element was assigned the average Young's modulus for that element. The finite element stresses compared favorably with the exact solution.[7]

Several other geometric shapes were analyzed and the results checked (within an acceptable error) wherever possible with exact solutions and acceptable approximate (thin-section) techniques.

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Turbulent Kinetic Energy Equation for a Transpired Turbulent Boundary Layer

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A FEW years ago Townsend[1] presented a discussion of several equilibrium turbulent boundary layers from the standpoint of the balance of the production and dissipation of turbulent kinetic energy in the various regions of the boundary layer. Among the cases considered was the particular case of a two-dimensional imcompressible turbulent boundary layer with uniform surface mass injection, which forms the subject of the present Note. Since that time, a considerable amount of work has been done on transpired turbulent boundary layers (Stevenson,[2] Kendall,[3] AlSaji,[4] Simpson, Moffat, and Kays,[5] Dahm and Kendall,[6] and Stevenson[7]) but the approach offered by Townsend has apparently not been exploited to date in studies of the inner region of an incompressible transpired turbulent boundary layer.

This Note presents experimental results correlated on the basis of the turbulent kinetic energy equation. These results indicate that the use of the turbulent kinetic energy equation provides additional information concerning the behavior of the turbulent boundary layer with surface mass injection.

The case considered is that of a uniform two-dimensional incompressible turbulent boundary layer developed initially over an impermeable surface with mass injected uniformly over the surface into the boundary layer at a downstream station. The axial pressure gradient is considered zero. For this case, the turbulent kinetic energy equation may be written as (Townsend[1] and Bradshaw et al.[8])

$$\langle\rho\rangle\langle U\rangle\partial(\langle q^2\rangle/2)/\partial x + \langle\rho\rangle\langle V\rangle\partial(\langle q^2\rangle/2)/\partial y + \langle\rho uv\rangle\partial\langle U\rangle/\partial y + \partial(\langle pv\rangle + \langle\rho q^2v\rangle/2)/\partial y + \langle\rho\rangle\epsilon = 0 \quad (1)$$

where $\langle U\rangle$ and $\langle V\rangle$ are the mean velocities in the x and y directions, respectively, $\langle q^2\rangle/2 = (\langle u^2\rangle + \langle v^2\rangle + \langle w^2\rangle)/2$ is the turbulent kinetic energy, p is the pressure fluctuation, $\langle\rho\rangle$ is the mean density, u, v, and w are the velocity fluctuations, and ϵ is the dissipation of the turbulent kinetic energy due to viscous effects.

Equation (1) represents the rate of change of the turbulent kinetic energy as the net sum of the convection, production, diffusion, and viscous dissipation of the turbulent kinetic energy.

The incompressible turbulent kinetic energy equation may be converted into a shear stress equation by defining[1,8]

$$\tau/\langle\rho\rangle = -\langle\rho uv\rangle/\langle\rho\rangle \quad (2)$$

$$a_1 = \tau/(\langle\rho\rangle\langle q^2\rangle) \quad (3)$$

$$L = (\tau/\langle\rho\rangle)^{3/2}/\epsilon \quad (4)$$

and

$$(\langle pv\rangle/\langle\rho\rangle + \langle\rho q^2v\rangle/2\langle\rho\rangle) = -a_2(\langle q^2\rangle^{3/2})\text{sign}(\partial\langle q^2\rangle/\partial y) \quad (5)$$

In these expressions a_1 and a_2 are taken as constants and L is an energy dissipation length. We should note that Townsend[1] introduced the expression used to model the diffusion term [Eq. (5)] from arguments concerning the structural equilibrium of the turbulence together with the sign of the gradient of the turbulent kinetic energy term. Bradshaw

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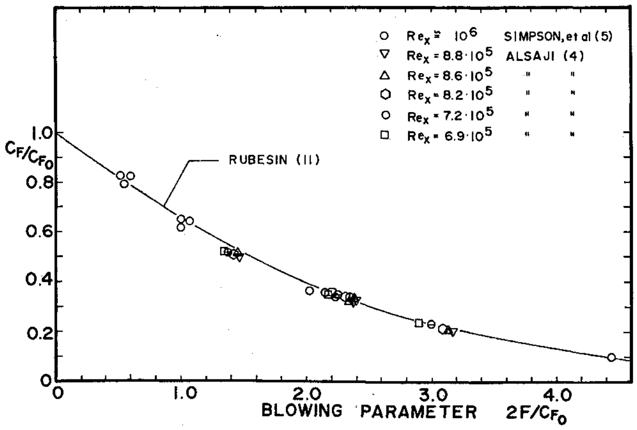

Fig. 1 Comparison of experimental data with Rubesin's theory, uniform injection.

has written the diffusion term including a factor which depends upon the maximum shear stress in the boundary layer. We have used Townsend's form in this study since the maximum shear stress in the boundary layer with mass transfer is not known without prior measurement of the shear stress distribution or a complete numerical mapping of the boundary-layer behavior.

The advantages in using the turbulent kinetic energy equation for the analysis of the turbulent boundary layer with surface mass injection may be shown by the following analysis. Utilizing Couette flow approximations and substituting Eqs. (2–5) into Eq. (1), the turbulent kinetic energy equation is converted into a shear stress equation which may be written as

$$[1/(\tau/\rho)^{1/2}]dU - (V_w/2a_1)[1/(\tau/\rho)^{3/2}]d(\tau/\rho) - [1/(2\tau/\rho)]d(\tau/\rho) = (1/L)dy \quad (6)$$

where all of the brackets denoting average quantities have been dropped.

In obtaining this expression the factor $3a_2/2(a_1)^{3/2}$ has been set equal to $\frac{1}{2}$, following the arguments of Townsend.[1] It should be noted that this approximation is based upon a limited amount of information and for more exact treatments; a more precise value of this factor should be determined. It should also be noted that the approximation $V = V_w$ has been used since the density is constant across the boundary layer.

For the two-dimensional turbulent boundary layer with surface mass injection, the mean momentum equation may be written as

$$\tau/\rho = u_\tau^2(1 + v_w^+u^+) \quad (7)$$

where

$$u_\tau^2 = \tau_w/\rho \quad (8)$$

$$v_w^+ = V_w/u_\tau \quad (9)$$

and

$$u^+ = U/u_\tau \quad (10)$$

Substituting these expressions into Eq. (6) and integrating yields

$$(2/v_w^+)[(1 + v_w^+u^+)^{1/2} - 1] + (v_w^+/a_1)(1 + v_w^+u^+)^{-1/2} - (\tfrac{1}{2})\ln(1 + v_w^+u^+) = \phi(v_w^+,u^+) = \int dy/L + B \quad (11)$$

where $B = d - 2/v_w^+$ and d is a constant of integration. This expression reduces to Stevenson's law of the wall for mass transfer[2] when the convection and diffusion terms are dropped from the turbulent kinetic energy equation and also reduces to the basic law of the wall when the surface mass injection goes to zero. If Eq. (11) is accurate, then the term $B = d - 2/v_w^+$ should be independent of blowing and L should be a linear function of distance from the wall, at least in the inner region of the boundary layer.

A subsonic wind tunnel similar to the one used by Wooldridge and Muzzy[9] was used to provide the boundary layer for experimental verification of Eq. (11). This wind tunnel has a test section 7 in. × 7 in. in cross section and 8 ft in length. The last two feet of the upper wall of the test section are fitted with an inconel porous plate through which air was injected. The freestream velocity was 25 fps with the pressure gradient through the last four feet of the test section set at approximately zero through the use of an adjustable bottom wall of the wind tunnel.

The performance of this facility has been thoroughly examined by AlSaji.[4] He obtained many velocity profiles through the test section for both the nonblowing case and the case with surface mass injection and found excellent agreement in evaluating the surface shearing stress by the momentum integral technique,[5] the method of Black and Sarnecki,[9] and Stevenson's method,[10] at least up to a blowing rate of $F = \rho_w V_w/\rho_e U_e = 4.78 \times 10^{-3}$. For a blowing rate of $F = 6.26 \times 10^{-3}$, considerable deviation between the momentum integral method and the methods of Stevenson and Black and Sarnecki was observed with the latter methods considered more reliable in the prediction of the surface shear stress. Figure 1 presents a comparison of the results obtained by AlSaji[4] for the wall shearing stress coefficient for blowing, normalized with the nonblowing value, compared with Rubesin's theory[11] and with the results presented by Simpson, Moffat, and Kays.[5] It appears from these results that Rubesin's method provides a reliable method for computing the effects of blowing on surface shear stress and also that the results obtained from the facility used for this study are at least consistent with another qualified facility.

In the study reported in this Note, additional velocity profiles were obtained for a series of surface injection rates with a hot-film boundary-layer probe using a Thermo-Systems Inc. model 1050 constant temperature anemometer and a model 1055 linearizer. A micromanometer with an accuracy to 0.0005 in. of water was used as a calibration standard.

These profiles were obtained at a point 85.5 in. from the entrance to the test section ($Re_x = 9 \times 10^5$). This location was also approximately 58 in. from the virtual origin of the turbulent boundary layer ($Re_{xv0} = 6.1 \times 10^5$) as determined by extrapolating the displacement thickness variations presented by AlSaji[4] to zero.

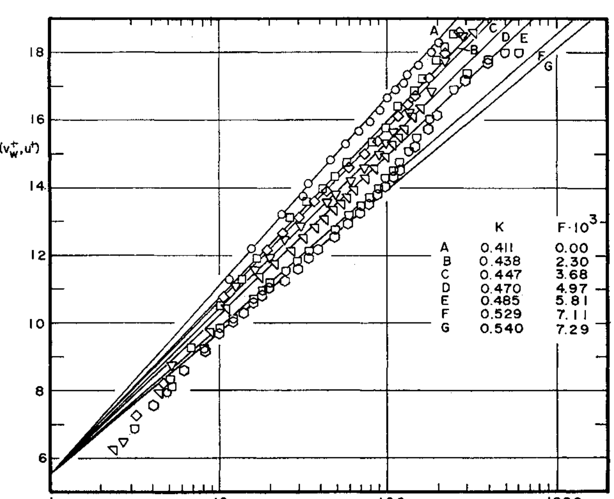

Fig. 2 Correlation of the turbulent kinetic energy equation across a turbulent boundary layer with mass injection at the wall. Re_x based on distance to the leading edge of the turbulent boundary layer.

The surface shear stress coefficient for nonblowing C_{f0} was determined in this study through the use of the following relationship presented by Schlichting[13]

$$C_{f0} = (2 \log R_x - 0.65)^{-2.3} \qquad (12)$$

where R_x is the freestream Reynolds number based upon an appropriate length. When the length Reynolds number was calculated using the distance to the virtual origin, the resulting value of C_{f0}, 0.00407, compared quite well with the values obtained by AlSaji[4] using Bradshaw's method[12] and the momentum integral method. However, when the Reynolds number was based upon the length from the entrance to the test section where the surface has been artificially roughened to promote transition, the value of the surface shear stress coefficient, $C_{f0} = 0.00381$, did not agree as well.

Once the value of C_{f0} was determined, the value of the surface shear stress coefficient with blowing C_f was obtained from Rubesin's method, Fig. 1, for a particular value of the blowing parameter $2F/C_{f0}$.

Figure 2 presents the left-hand side of Eq. (11) plotted vs log yu_τ/ν for a series of increasing injection rates. In the evaluation of the surface friction velocity, $u_\tau^2 = U_e^2 C_f/2$, the length Reynolds number was calculated using the distance from the entrance to the test section to the station where the profiles were obtained. Figure 3 presents the left-hand side of Eq. (11) plotted vs log yu_τ/ν for the same series of injection rates but with the Reynolds number calculated using the distance from the virtual origin of the turbulent boundary layer to the station where the velocity profiles were obtained. The constant a_1 was taken as 0.15 for the evaluation of these figures.

It is apparent from the results indicated in Figs. 2 and 3 that the correlation of the turbulent kinetic energy equation across the turbulent boundary layer with mass injection yields a constant value of $B = (d - 2/v_w^+) = 5.5$ for the blowing rates used and that the slope $(1/K)$ of the logarithmic portion of the velocity function is dependent upon the magnitude of the surface injection rate.

It is interesting to note that the value of K for the no-blowing case was found to be 0.411 when the surface shearing stress was based upon the distance from the entrance of the wind tunnel. However, when the surface shearing stress is based on the length to the virtual origin, the resulting plot produces a value of $K = 0.433$. Independent computation of the value of C_{f0} using the velocity profiles at the point in question yielded a value of C_{f0} much closer to the virtual origin approach than by using the length to the leading edge of the test section.

When the value of K is plotted vs the mass injection parameter F, an approximate linear function results. This expression may be represented as

$$K = 14.25F + 0.411 \text{ (leading edge Reynolds number)}$$
$$K = 11.50F + 0.433 \text{ (virtual origin Reynolds number)}$$

The relationship of K to the characteristics of turbulence in a variety of boundary layer environments is presently under investigation in our laboratory.

From the results presented in Figs. 2 and 3 it is apparent that the effects of surface mass injection are reflected across a considerable portion of the boundary layer and are not confined to effects similar to those introduced by surface roughness. It also appears from these results that the B term included on the right-hand side of Eq. (11) is independent of the blowing rate. Neglect of the convection and diffusion terms in the turbulent kinetic energy equation near the wall would produce an apparent dependency on blowing rate in the B term.

Our use of the a_1 and a_2 terms as constants, obtained from no-blowing boundary layers, was based upon the assumption that these terms are functions of the basic turbulence and that the modeling functions should still be closely approximated in turbulent boundary layers with surface mass injection. We are presently carrying out a program of research designed to obtain the necessary statistical properties of the boundary layer required to verify these relationships.

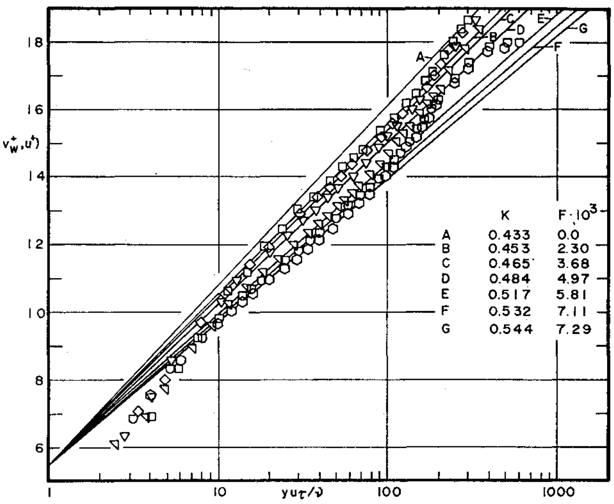

Fig. 3 Correlation of the turbulent kinetic energy equation across a turbulent boundary layer with mass injection at the wall. Re_{xv_0} based on distance to the virtual origin of the turbulent boundary layer.

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